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Skyrmions on an elastic cylinder

R. Dandoloff[1,a] and A. Saxena[2]

[1] Laboratoire de Physique Théorique et Modélisation, Université de Cergy-Pontoise, 5 Mail Gay-Lussac, 95031 Cergy-Pontoise Cedex, France
[2] Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

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Abstract. For a spin-polarized electron gas on an elastic cylinder in an external axial magnetic field and an axial electric field we find that the corresponding Euler-Lagrange equation is the double sine-Gordon (DSG) equation with an exact 2π-skyrmion solution. The DSG skyrmion is stabilized, without Coulomb repulsion, by the curvature of the cylinder. It adopts a characteristic length ξ which is smaller than the radius of the cylinder. For an elastic cylinder this mismatch of length scales causes a deformation of the cylinder in the region of the skyrmion.

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There are many mesoscopic physical systems that exemplify an interplay between geometry and nonlinearity. A spin-polarized electron gas on a mesoscopic cylinder is an important representative of such a system. From this perspective we note that a planar two dimensional electron system (2DES) in a strong magnetic field $\mathbf{B}$ behaves like a (quantum Hall) ferromagnet [1]. Typical low energy topological excitations in this system are specific spin textures known as skyrmions [2]. In order to stabilize skyrmions in 2D one must consider electron-electron (Coulomb) repulsion. The competition between the Zeeman and Coulomb energies leads to a finite characteristic length of the skyrmions in a *flat* 2DES. Our goal is to study a 2DES on a *curved* surface, specifically a cylinder, because the underlying geometry (*i.e.* the cylinder) naturally provides a characteristic length scale ρ_0, the radius of the cylinder, which may stabilize the skyrmions *without* the Coulomb repulsion.

We consider the (field theoretic) Hamiltonian of Sondhi *et al.* [1] and look for stable ground state configurations of the order parameter ($\hat{n}$, $\hat{n}^2 = 1$). Next, we consider the 2DES on a cylinder with an applied axial magnetic field $\mathbf{B}$. Furthermore, we are interested only in solutions with cylindrical symmetry. If we introduce cylindrical coordinates (ρ, ϕ, z) and if we restrict the order parameter to lie on the surface of a unit sphere

$$\mathbf{n} = (\sin\theta\cos\Phi, \sin\theta\sin\Phi, \cos\theta)$$

then we are interested in solutions with $\Phi = \phi$ and $\theta = \theta(z)$ [3,4]. The Hamiltonian now reads:

$$H = 2\pi\rho_0 \int_{-\infty}^{\infty} \mathrm{d}z[\alpha' \left(\theta_z^2 + \frac{\sin^2\theta}{\rho_0^2}\right) + B_0(1-\cos\theta)], \quad (1)$$

where $B_0 = g\mu B\overline{\sigma}$, $\overline{\sigma}$ is the average electronic density, μ is the magnetic moment and g denotes the g factor of the electrons in the magnetic material of the cylinder. Here α' is the spin stiffness modified by the Coulomb repulsion.

In principle, the Coulomb repulsion energy should also be included:

$$H_{coulomb} = \frac{(2\pi\rho_0)^2}{2} \int\int_{-\infty}^{\infty} \mathrm{d}z\mathrm{d}z' V(z-z')q(z)q(z')], \quad (2)$$

where $q(z) = (\sin\theta/\rho)(\mathrm{d}\theta/\mathrm{d}z)$ is the Pontryagin (or topological charge) density and $V(z-z')$ is the modified (*i.e.* screened) Coulomb potential between the electronic densities $q(z)$. This interaction is not required to stabilize the skyrmions here unlike in the 2D flat case, where it introduces a characteristic length scale. The skyrmions on a cylinder are stabilized by adopting their characteristic length from the curvature of the cylinder [3,4]. In the present case the Coulomb interaction would merely modify this characteristic length, which may be effectively accounted for by modifying the spin stiffness (akin to the one dimensional spin-phonon coupled systems). This will become apparent below and hence forth we will neglect the Coulomb interaction in our discussion.

If in addition, we apply an external axial electrical field $\mathbf{E}$, the Hamiltonian density in equation (1) will be

[a] e-mail: rossen.dandoloff@ptm.u-cergy.fr

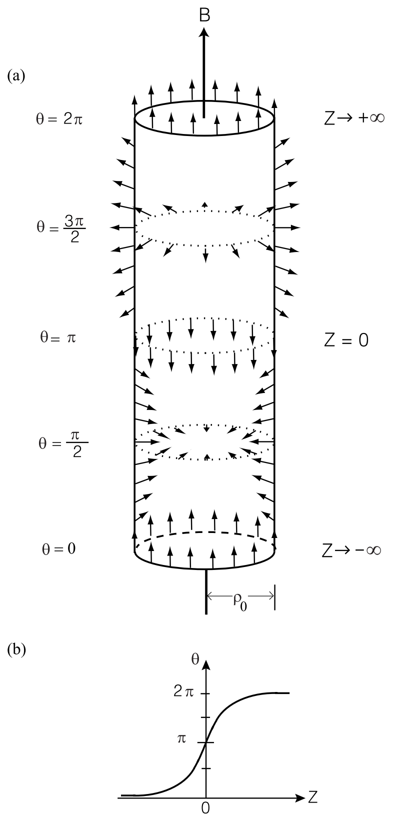

Fig. 1. Cylindrically symmetric $0 \to 2\pi$ 2-skyrmion on an infinite cylinder in the presence of both an axial electric and an axial magnetic field.

modified by adding a term $q(z)E$. We are now left with the Hamiltonian density h:

$$h = \alpha' \left(\theta_z^2 + \frac{\sin^2 \theta}{\rho_0^2} \right) + g\mu B\overline{\sigma}(1 - \cos\theta) + q(z)E. \quad (3)$$

We look for spin textures that minimize this Hamiltonian. Notice that the term

$$q(z)E = \frac{\sin\theta}{\rho_0} \frac{\mathrm{d}\theta}{\mathrm{d}z} E = -\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{E}{\rho_0} \cos\theta \right) \quad (4)$$

is a total derivative with respect to z and as a consequence it does not contribute to the Euler-Lagrange (EL) equation. In other words, the electric field does not affect the charge density $q(z)$ in this geometry. This leaves us with the following EL equation:

$$2\theta_{zz} = \frac{1}{\rho_0^2} \sin 2\theta + \frac{1}{\rho_B^2} \sin\theta. \quad (5)$$

Here $\rho_B = \sqrt{\alpha'/g\mu B\overline{\sigma}}$ is a magnetic length.

The EL equation, equation (5), is the well-known double sine-Gordon (DSG) equation [5]. For the cylindrical configuration with axial electric and magnetic fields it has an exact solution for a single 2-skyrmion (*i.e.*, Pontryagin index two [6]), see Figure 1,

$$\theta(z) = 2\tan^{-1} \left(\frac{\rho_B}{\xi \sinh \frac{z}{\xi}} \right), \quad \xi = \frac{\rho_0 \rho_B}{(\rho_0^2 + \rho_B^2)^{1/2}}, \quad (6)$$

where ξ is the characteristic length of the 2π-skyrmion. Equivalently, $\xi = \sqrt{\rho_0 \rho_B}\sqrt{\rho_0 \rho_B/(\rho_0^2 + \rho_B^2)}$.

This 2π-skyrmion has a finite energy because at both $+\infty$ and $-\infty$ the spins are parallel to the external magnetic field **B** (Fig. 1). The highest contribution to the magnetic energy density, due to the interaction between the spins and the external magnetic field comes from the sector between the centers of the two skyrmions: there the spins are opposite to the external magnetic field. Therefore the whole 2π-skyrmion would like to collapse and eliminate the region where the spins are opposite to the magnetic field. However, the collapse of the 2π-skyrmion would leave a π-skyrmion with infinite energy, because *e.g.* at $+\infty$ the spins will be oriented against the external magnetic field. This means that there is *de facto* hard-core repulsion between the two π-skyrmions in the system by virtue of the curvature of the cylinder; for more details see [9]. (Therefore it is unnecessary to introduce an additional hard-core repulsion term, as this is done in the case of a plane 2D problem [1].)

The skyrmion chooses a characteristic length which is smaller than the geometric mean of the cylinder radius and the magnetic length (see Eq. (6)). With increasing magnetic field the characteristic length of the skyrmion decreases. The energy of the 2-skyrmion

$$E_s = 16\pi J \left\{ \left(1 + \frac{\rho_0^2}{\rho_B^2} \right)^{1/2} + \frac{\rho_0^2}{\rho_B^2} \sinh^{-1} \frac{\rho_B}{\rho_0} \right\} \quad (7)$$

is larger than $16\pi J$ (the mimimum energy for the homotopy class [2,7] with winding number $Q = 2$). The additional energy cost is due to the repulsive interaction between the two skyrmions in a 2-skyrmion. Only in the limit $\rho_B \to \infty$ ($B \to 0$) we get $E_s \to 16\pi J$ and $\xi \to \rho_0$. In addition, there is a 2-skyrmion lattice solution for $\rho_0 \geq \rho_B$:

$$\tan \frac{\theta_L(z)}{2} = \pm \frac{\rho_B}{\xi_L} \frac{\mathrm{cn}\left(\frac{z - z_0}{\xi_L}, k \right)}{\mathrm{sn}\left(\frac{x - x_0}{\xi_L}, k \right)}, \quad (8)$$

with periodicity $d = 4K\xi_L$, where $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$ are Jacobi elliptic functions and $K(k)$ is the complete elliptic integral of the first kind [8]. Here

$$\xi_L^2 = \frac{\rho_B^2}{\tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_2}{2}}, \quad k^2 = \frac{\tan^2 \frac{\theta_2}{2}}{\tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_2}{2}}, \quad (9)$$

and $\tan^2 \frac{\theta_1}{2}$ and $\tan^2 \frac{\theta_2}{2}$ are the two non-zero roots of:

$$\left(1 - \frac{\rho_B^2 V_0}{2} \right) \tan^4 \frac{\theta}{2} + \rho_B^2 \left(\frac{1}{\xi^2} - V_0 \right) \tan^2 \frac{\theta}{2} - \frac{\rho_B^2 V_0}{2} = 0, \quad (10)$$

with $0 \leq V_0 < 2/\rho_B^2$. Here $V_0 = 0$ and $V_0 = 2/\rho_B^2$ correspond to $k = 1$ (single 2-skyrmion case) and $k = 0$, respectively. The energy of the 2-skyrmion lattice is given in reference [9].

The spins in the center of the 2-skyrmion are opposed to the direction of the magnetic field. Therefore, the Zeeman energy ($\mathbf{n}{\cdot}\mathbf{B}$) tries to bring the two parts of the 2-skyrmion together in order to reduce the area on the cylinder between the two parts. The Coulomb term that we neglected would oppose the collapse of the 2-skyrmion but the main stabilizing effect on the size of the skyrmion is due to the geometric support: the cylinder itself.

The main conclusion is that the skyrmions on a cylinder are stabilized by the interplay of the Zeeman energy *via* the magnetic length and the geometric length ρ_0 of the cylinder.

Let us now consider an elastic cylinder, that is, we relax the constraint that the radius of the cylinder is fixed at ρ_0. Instead we now have $\rho = \rho(z)$. Solutions given in reference [9] apply here and we find that the cylinder deforms in the region of the 2-skyrmion. There are further consequences of this effect. The shrunk area of the cylinder may create *localized* electronic states [10] and this will localize the skyrmions.

There are other interesting possibilities. A smaller radius means higher Landau level energy $\epsilon_n \sim (1/\rho^2)$ which implies that electrons would leave this area and the central ring would become positively charged creating an axial dipole (For time-dependent solutions, an oscillating dipole emits radiation–the cylinder will act like a nano-antenna!). For a skyrmion lattice solution, equation (8), this will lead to alternating +ve and −ve radial charge stripes. Instead of the magnetoelastic coupling, we can also envision a shrinking of the cylinder arising from electron-phonon interaction. The skyrmion should have characteristic signatures in the photoluminescence spectra (different from their photoluminescence spectra of a flat 2DES [11]). Similarly, the deformation of the cylinder should have signatures in the Raman spectra as well as in the phase shift and attenuation of an ultrasonic pulse [12] traveling along the deformed cylinder. Finally, it would be interesting to study skyrmion stabilization on other simple curved geometries such as a sphere and a torus. The latter introduces two geometric length scales in addition to a nontrivial genus-1 topology.

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